On the existence of boundary blow-up solutions for a general class of quasilinear elliptic systems

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Abstract. In this paper, we investigate the following quasilinear elliptic system $(P)$ with explosive boundary conditions:

\[
\begin{align*}
\Delta_p u &= f_1(x, u, v) \quad \text{in } \Omega; \quad u|_{\partial \Omega} = +\infty, \quad u > 0 \quad \text{in } \Omega, \\
\Delta_q v &= f_2(x, u, v) \quad \text{in } \Omega; \quad v|_{\partial \Omega} = +\infty, \quad v > 0 \quad \text{in } \Omega,
\end{align*}
\]

where $\Omega$ is a smooth bounded domain of $\mathbb{R}^N$, $N \geq 2$, $1 < p, q < +\infty$ and $f_1, f_2$ are two Carathéodory functions in $\Omega \times (\mathbb{R}_+^* \times \mathbb{R}_+^*)$. Under rather general conditions on $f_1$ and $f_2$ and assuming the existence of a sub and supersolutions pair, we prove the existence of a large solution to $(P)$. Next, we give some applications in Biology.

1 Introduction

1.1 Presentation of the problem

In this paper we are interested in the following boundary blow-up quasilinear elliptic system:

\[
(P) \quad \begin{cases}
\Delta_p u = f_1(x, u, v) \quad \text{in } \Omega; \quad u|_{\partial \Omega} = +\infty, \quad u > 0 \quad \text{in } \Omega, \\
\Delta_q v = f_2(x, u, v) \quad \text{in } \Omega; \quad v|_{\partial \Omega} = +\infty, \quad v > 0 \quad \text{in } \Omega.
\end{cases}
\]

Here, $\Omega$ is a bounded domain of $\mathbb{R}^N$, $N \geq 2$, with a $C^{1,\alpha}$ boundary $\partial \Omega$, for some $0 < \alpha < 1$. In the left-hand sides of the two partial differential equations, $1 < p, q < +\infty$ and

\[
\Delta_r w \overset{\text{def}}{=} \text{div}(|\nabla w|^{r-2}\nabla w)
\]

denotes the $r$-Laplace operator. And in the right-hand sides, $f_1, f_2 : \Omega \times (\mathbb{R}_+^* \times \mathbb{R}_+^*) \to \mathbb{R}$ satisfy several assumptions we detail later. We define a boundary blow-up solution to $(P)$ in the following sense:

Definition 1.1. A couple $(u^*, v^*) \in [W^{1,p}_{\text{loc}}(\Omega) \cap C(\Omega)] \times [W^{1,q}_{\text{loc}}(\Omega) \cap C(\Omega)]$ is said to be a weak positive boundary blow-up solutions pair (or just a large solution for
short) to problem \( (P) \) if both \( u^* \) and \( v^* \) are positive in \( \Omega \), if

\[
\lim_{d(x) \to 0^+} u^*(x) = \lim_{d(x) \to 0^+} v^*(x) = +\infty,
\]

where \( d(x) \) is the distance from any point \( x \in \Omega \) to the boundary \( \partial \Omega \), and if equations

\[
\Delta_p u^* = f_1(x, u^*, v^*) \quad \text{and} \quad \Delta_q v^* = f_2(x, u^*, v^*) \quad \text{in} \quad \Omega
\]

are satisfied in the distribution sense; \( i.e. \), for any couple of test functions \( (\varphi, \psi) \in C^\infty_c(\Omega) \times C^\infty_c(\Omega) \),

\[
\int_\Omega |\nabla u^*|^{p-2} \nabla u^* \cdot \nabla \varphi \, dx = \int_\Omega f_1(x, u^*, v^*) \varphi \, dx
\]

and

\[
\int_\Omega |\nabla v^*|^{q-2} \nabla v^* \cdot \nabla \psi \, dx = \int_\Omega f_2(x, u^*, v^*) \psi \, dx.
\]

Boundary blow-up solutions in different competitive, cooperative or predator/prey systems may arise in population dynamics problems with "spacial heterogeneity". Namely, in these type of non-homogeneous systems, one of the equations contains an absorption term with a multiplying weight which vanishes in a subset \( \omega \) of the domain \( \Omega \). This traduces that the growth of the associated specie, may be "out of control" in \( \omega \) and a blow-up phenomenon may occur at the boundary \( \partial \omega \). We refer to [5, 9, 10, 29] for more details about the modelling. We may also cite here [7] where an other type of boundary blow-up elliptic system, related to a fluid dynamics model, is investigated. But, this problem involves a gradient dependency in the non-linearity of one of the equations, which is not exactly the subject of this paper.

One of the main difficulties of boundary blow-up problems relies on the \textit{a priori} lack of knowledge of the precise blow-up behaviour of the solutions near the boundary. Then, the direct use of comparison principles or Sobolev spaces (even weighted), in the whole domain \( \Omega \), is inappropriate in this context. Despite this, sub and supersolution methods may be employed to study some particular boundary blow-up elliptic systems. We may cite here for instance [6, 12, 15, 16, 22, 27, 37], for the semilinear case \( i.e. \ p = q = 2 \) and most recently [3, 13, 23, 36, 38], where the quasilinear case is investigated.

In the above cited papers, the authors take advantage of the particular nature (competitive, cooperative or predator/prey) of the system they considered to construct large solutions. Namely, the nature of the system is related to the monotonicity of the right-hand sides \( f_1 \) with respect to \( v \) and \( f_2 \) with respect to \( u \). Then,
from an explicit sub and supersolutions pair, they use a monotone iterative scheme technique which provides the existence of a large solution to the system.

However, in these previous mentioned studies, the definition of sub and supersolutions used and the monotonicity method employed seem different with respect to the competitive, the cooperative or the predator/prey configuration. Nevertheless, to our knowledge, no general existence theorem, i.e. which does not take into account of the nature of the system, has been written yet for boundary blow-up systems. Moreover, it is also obvious that all biological or chemical interactions cannot be model only by one of the three above configurations. For instance, some Gierer-Meinhardt elliptic systems (see [19]) contain both a competitive and a cooperative contribution in each right-hand side, which results in a sign-changing behaviour of $f_1$ and $f_2$. Nevertheless, the previous sub and supersolutions methods used cannot easily be generalized for this type of systems.

For these two reasons, in this paper, our goal is to develop a new sub and supersolution method allowing to study the most general class of boundary blow-up systems shaped as $(P)$ as possible; in particular encompassing all the above mentioned type of systems (competitive, cooperative, predator/prey and with "sign-changing" right-hand sides). For this, we want to get read of any monotonicity or sign assumption on $f_1$ and $f_2$. Then, we only make in this study the following quite non-restrictive hypotheses on the right-hand sides:

<table>
<thead>
<tr>
<th>Hypothesis</th>
<th>Description</th>
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<tbody>
<tr>
<td>$(H_1)$</td>
<td>$f_1$ and $f_2$ are two Carathéodory functions on $\Omega \times (\mathbb{R}<em>+^* \times \mathbb{R}</em>+^*)$.</td>
</tr>
<tr>
<td>$(H_2)$</td>
<td>For any $(t_1, t_2) \in \mathbb{R}<em>+^* \times \mathbb{R}</em>+^*$, $f_1(\cdot, t_1, t_2)$ and $f_2(\cdot, t_1, t_2)$ are $L_\text{loc}^\infty(\Omega)$.</td>
</tr>
<tr>
<td>$(H_3)$</td>
<td>For a.e. $x \in \Omega$, $f_1(x, \cdot, \cdot)$ and $f_2(x, \cdot, \cdot)$ are of class $C^1(\mathbb{R}<em>+^* \times \mathbb{R}</em>+^*)$.</td>
</tr>
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1.2 Contents of the paper

The main result of this paper, Theorem 2.1, is given in section 2. From the knowledge of a sub and supersolutions pair, it establishes the existence of a large solution to $(P)$ under an additional growth assumption on $f_1$ and $f_2$. To overcome the difficulties generated by the lack of monotonicity of the right-hand sides, we first generalise the definition of sub and supersolutions pairs to $(P)$ (see Definition 1.2 below). Then, we combine a fixed point approach based on an idea developed in [17, 21] with some "interior considerations" extracted from [13, 15].

In a second time, we illustrate the contribution of our main result. For that, we give in section 3 a first example of application considering a boundary blow-up
system shaped as

\[
\begin{cases}
\Delta_p u = K_1(x) u^{a_1} v^{b_1} \text{ in } \Omega; \\
\Delta_q v = K_2(x) v^{a_2} u^{b_2} \text{ in } \Omega;
\end{cases}
\]

\[ u|_{\partial \Omega} = +\infty, \quad u > 0 \text{ in } \Omega, \]

\[ v|_{\partial \Omega} = +\infty, \quad v > 0 \text{ in } \Omega. \]  

(1.1)

The choice to focus on this particular class of quasilinear systems is motivated by several previous publications (see [3, 13, 23, 36, 38]) where (1.1) is studied throughout different configurations. Namely, under the standard super-homogeneity condition

\[(a_1 - p + 1)(a_2 - q + 1) - b_1 b_2 > 0, \]  

(1.2)

the authors give the same kind of results on the existence/non-existence, the uniqueness and the asymptotic behaviour of solutions to (1.1) in the competitive or in the cooperative case. More precisely, in [13, 38] the competitive case (i.e. with \(b_1, b_2 > 0\)) is investigated. Whereas in [3, 36] the authors considered the cooperative case (i.e. with \(b_1, b_2 < 0\)). Accurately, [3] starts to study (1.1) without any weight \(K_i\), for \(i = 1, 2\) and [36] gives a generalization considering weights behaving like a power of distance function. Namely, for some \(\lambda_1 < p\) and \(\lambda_2 < q\),

\[
\lim_{d(x) \to 0^+} d(x)^{\lambda_1} K_1(x) = 1 \quad \text{and} \quad \lim_{d(x) \to 0^+} d(x)^{\lambda_2} K_2(x) = 1. \]  

(1.3)

Concerning problem (1.1), we bring here two contributions:

(i) The first one is that, we do not impose any sign assumption on \(b_1\) and \(b_2\), thus we study the competitive, cooperative and predator/prey (i.e. with \(b_1 b_2 < 0\)) cases in the same way. For that, condition (1.2) is naturally replaced by

\[(a_1 - p + 1)(a_2 - q + 1) - |b_1 b_2| > 0.\]

To our knowledge, this is the first time that the predator/prey case is investigated for problem (1.1).

(ii) The second one relies on the adding of Karamata perturbation in assumption (1.3) on the weights \(K_i(x)\). This makes the computations harder but allows to widen the class of systems which may be considered.

Then, under appropriate assumptions on \(K_1, K_2\) and exponents \(a_1, a_2, b_1\) and \(b_2\), we prove in Theorem 3.5 both the existence and the precise behaviour of large solutions to (1.1).

Actually, Theorem 3.5 is a direct application of Theorem 2.1 which relies on the validity of the existence of a suitable sub and supersolutions pair to (1.1). For
this, we will first make the study of a connected boundary blow-up scalar elliptic equation shaped as

\[
\begin{cases}
    \Delta_r w = K(x) w^\delta, & w > 0 \quad \text{in } \Omega, \\
    w|_{\partial\Omega} = +\infty.
\end{cases}
\]  

(1.4)

Precisely, under suitable assumptions on the parameters, we establish in Proposition 3.3 (whose proof is given in Appendix A) the existence and the precise behaviour of a large solution to (1.4).

Concerning the boundary blow-up problem (1.4), we refer to [14] where a detailed history of this equation, containing a list of the main references on the topic, is presented. We can add to this list [30] (and references therein), where an alternative method to the sub and super solutions method is developed via Keller-Osserman methods (see [26, 32]); and also [1], where one of the first studies (with [2]) on a boundary blow-up equation containing Karamata perturbation is made.

Finally, in section 4, we although give in Theorems 4.1 and 4.2 two other illustrations of Theorem 2.1 throughout the studies of systems presenting sign-changing right-hand sides.

Note that in this paper, we only focus on the existence of a large solution to (P). For this reason, the (although interesting) questions of non-existence or uniqueness of solutions to problems (1.1) and (1.4) will not be raised here (see points (iii) in Remarks 3.4 and 3.6 for more details).

1.3 Notations

Throughout this paper, we will use the following notations and definitions:

(i) For \( x \in \Omega \), we denote the distance from \( x \) to \( \partial\Omega \) by

\[
d(x) \overset{\text{def}}{=} d(x, \partial\Omega) = \inf_{y \in \partial\Omega} d(x, y).
\]

(ii) We denote by \( D \overset{\text{def}}{=} \sup_{x,y \in \Omega} d(x, y) \), the diameter of the domain \( \Omega \).

(iii) Let \( f, g \in C(\Omega) \) be two positive functions in \( \Omega \). Then, we write

\[
f(x) \approx g(x) \quad \text{in } \Omega
\]

if there exist \( C_1, C_2 > 0 \) such that \( C_1 g \leq f \leq C_2 g \) in \( \Omega \).
(iv) Let $1 < r < +\infty$. We denote by $\varphi_{1,r} \in W_{0}^{1,r}(\Omega)$, the (unique) positive and $L^{r}$-renormalized eigenfunction corresponding to the first eigenvalue of the $r$-Laplace operator:

$$\lambda_{1,r} \overset{\text{def}}{=} \inf \left\{ ||\nabla v||_{L^{r}(\Omega)} \in \mathbb{R}, \ v \in W_{0}^{1,r}(\Omega) \text{ and } ||v||_{L^{r}(\Omega)} = 1 \right\}$$

(see [33, §1.4 - p. 20]). By definition, it is the unique weak solution to the following eigenvalue problem:

$$\begin{cases}
-\Delta_{r} w = \lambda_{1,r} w^{r-1}, & w \geq 0 \text{ in } \Omega, \\
w_{|\partial \Omega} = 0, & ||w||_{L^{r}(\Omega)} = 1.
\end{cases}$$

We just recall that, from Moser iterations [31] and [28, Theorem 1], $\varphi_{1,r} \in C^{1,\beta}(\Omega)$, for some $0 < \beta < 1$, and from the strong maximum principle for quasilinear operators (see [35, Theorem 10]), $\varphi_{1,r}$ satisfies

$$\varphi_{1,r}(x) \approx d(x) \text{ in } \Omega$$

and

$$\varphi_{1,r}(x)^{r} + |\nabla \varphi_{1,r}(x)|^{r} \approx 1 \text{ in } \Omega. \quad (1.6)$$

(v) Let $\underline{w}, \overline{w} \in W^{1,r}_{\text{loc}}(\Omega) \cap C(\Omega)$ be two positive functions such that $\underline{w} \leq \overline{w}$ in $\Omega$. We define the convex set

$$[\underline{w}, \overline{w}] \overset{\text{def}}{=} \left\{ w \in W^{1,r}_{\text{loc}}(\Omega) \cap C(\Omega), \ w \leq w \leq \overline{w} \text{ in } \Omega \right\}.$$

**Definition 1.2.** Let $\underline{u}, \overline{u} \in W^{1,p}_{\text{loc}}(\Omega) \cap C(\Omega)$ and $\underline{v}, \overline{v} \in W^{1,q}_{\text{loc}}(\Omega) \cap C(\Omega)$ be four positive functions such that $\underline{u} \leq \overline{u}$ in $\Omega$, $\underline{v} \leq \overline{v}$ in $\Omega$ and

$$\lim_{d(x) \to 0^{+}} \underline{u}(x) = \lim_{d(x) \to 0^{+}} \overline{u}(x) = +\infty.$$ 

The couple $(\underline{u}, \underline{v})$ and $(\overline{u}, \overline{v})$ is said to be a *sub and supersolutions pair* to (P) if the following inequalities are satisfied in the distribution sense:

$$\Delta_{p} \underline{u} \geq f_{1}(x, \underline{u}, \underline{v}) \text{ in } \Omega, \text{ for any } v \in [\underline{v}, \overline{v}], \quad (1.7)$$
$$\Delta_{q} \underline{u} \geq f_{2}(x, \underline{u}, \underline{v}) \text{ in } \Omega, \text{ for any } u \in [\underline{u}, \overline{u}], \quad (1.8)$$
$$\Delta_{p} \overline{u} \leq f_{1}(x, \overline{u}, \overline{v}) \text{ in } \Omega, \text{ for any } v \in [\underline{v}, \overline{v}], \quad (1.9)$$
$$\Delta_{q} \overline{u} \leq f_{2}(x, \underline{u}, \overline{v}) \text{ in } \Omega, \text{ for any } u \in [\underline{u}, \overline{u}]. \quad (1.10)$$

Then, the conical shell $[\underline{u}, \overline{u}] \times [\underline{v}, \overline{v}]$ of sub and supersolutions is denoted by $C$. 
2 Main result

Theorem 2.1. Let \((u, v), (\bar{u}, \bar{v}) \in [W^{1,p}_{\text{loc}}(\Omega) \cap C(\Omega)] \times [W^{1,q}_{\text{loc}}(\Omega) \cap C(\Omega)]\) be a sub and supersolutions pair to \((P)\). Assume in addition that there exist \(g_1, g_2 \in L^\infty_{\text{loc}}(\Omega)\) such that:

\[\forall \text{ a.e.} \, x \in \Omega, \forall (u, v) \in C, \quad \frac{\partial f_1}{\partial u}(x, u, v) \leq g_1(x) \quad \text{and} \quad \frac{\partial f_2}{\partial v}(x, u, v) \leq g_2(x). \quad (2.1)\]

Then, there exists a large solution \((u^*, v^*) \in C\) to problem \((P)\). Moreover, \((u^*, v^*) \in C^1_{\text{loc}}(\Omega) \times C^1_{\text{loc}}(\Omega)\), for some \(0 < \beta < 1\).

Remark 2.2.

(i) In the semilinear case, if we assume that \(f_1, f_2\) are locally \(\beta\)-Hölder continuous with respect to \(x \in \Omega\), for some \(0 < \beta < 1\), the Schauder’s estimates (see for instance [20, Theorem 6.14]) ensure us that \((u, v) \in C^2_{\text{loc}}(\Omega) \times C^2_{\text{loc}}(\Omega)\). Thus, \((u, v)\) is a large solution to \((P)\) in the classical sense.

(ii) Instead of assumptions \((H_3)\) and (2.1), we can rather suppose that there exist \(\tilde{g}_1, \tilde{g}_2 \in L^\infty_{\text{loc}}(\Omega)\) and \(\varrho_1, \varrho_2 > 1\) such that: \(\forall \text{ a.e.} \, x \in \Omega, \forall (u, v) \in C,\)

\[w \mapsto f_1(x, w, v) - \tilde{g}_1(x)w^{\varrho_1-1} \text{ is non increasing on } [u, \bar{v}],\]

\[w \mapsto f_2(x, u, w) - \tilde{g}_2(x)w^{\varrho_2-1} \text{ is non increasing on } [v, \bar{v}].\]

In this case, we get the same result and this condition is sharper if \(\varrho_1, \varrho_2 > 2\). For that, it suffices to replace the first equation of problem \((P_1^\nu)\), given below, by

\[\Delta_p \tilde{u}_n + \tilde{h}_1(x, \tilde{u}_n) = f_1(x, u, v) + \tilde{g}_1(x)u^{\varrho_1-1} \text{ in } \Omega_n,\]

with \(\tilde{h}_1 : \Omega_n \times \mathbb{R} \to \mathbb{R}_+\) the cut-off function defined as follows:

\[\tilde{h}_1(x, z) \overset{\text{def}}{=} \begin{cases} 
\tilde{g}_1(x)\bar{v}^{\varrho_1-1} & \text{if } z \geq \bar{v}(x), \\
\tilde{g}_1(x)z^{\varrho_1-1} & \text{if } z \in [u(x), \bar{v}(x)], \\
0 & \text{if } z \leq u(x)
\end{cases} \tag{2.2}\]

and proceed similarly for \((P_2^\nu)\).

Proof. (of Theorem 2.1) We follow here the idea of [15, Theorem A.2] and [13, Theorem A.2] considering at first the system at the interior of \(\Omega\). For this, we consider \((\Omega_n)_{n \in \mathbb{N}^*}\) an increasing sequence (for the inclusion) of sub-domains of
\( \Omega \) with smooth boundaries (at least \( C^{1,\alpha} \)) such that \( \Omega_n \xrightarrow[n \to +\infty]{} \Omega \) in the Hausdorff topology with: \( \forall n \in \mathbb{N}^* \) (large enough), \( \forall x \in \Omega_n \),

\[
\frac{D}{n+1} \leq d(x, \partial \Omega) < \frac{D}{n}.
\]

And for any \( n \in \mathbb{N}^* \), we look for a weak solution to problem:

\[
(P^n) \begin{cases}
\Delta_p u_n = f_1(x, u_n, v_n) & \text{in } \Omega_n; \\
\Delta_q v_n = f_2(x, u_n, v_n) & \text{in } \Omega_n;
\end{cases}
\]

\( u_n |_{\partial \Omega_n} = u \), \( u > 0 \) in \( \Omega_n \),

\( v_n |_{\partial \Omega_n} = v \), \( v_n > 0 \) in \( \Omega_n \).

**Remark 2.3.** It is always possible to construct a such sequence \( (\Omega_n)_{n \in \mathbb{N}^*} \) because of the regularity we assume on the domain \( \Omega \) (see for instance [4, Theorem A.2 p.27]).

As stated in the introduction, we use here a fixed-point argument. Then, let \( n \in \mathbb{N}^* \) (large enough), let \((u, v)\) be a fixed couple of the conical shell \( \mathcal{C} \) of sub and supersolutions and let us consider problems \((P^n_1)\) and \((P^n_2)\) defined as follows:

\[
(P^n_1) \begin{cases}
\Delta_p \bar{u}_n - h_1(x, \bar{u}_n) = f_1(x, u, v) - g_1(x)u, & \bar{u}_n > 0 \text{ in } \Omega_n, \\
\bar{u}_n |_{\partial \Omega} = u,
\end{cases}
\]

\[
(P^n_2) \begin{cases}
\Delta_q \bar{v}_n - h_2(x, \bar{v}_n) = f_2(x, u, v) - g_2(x)v, & \bar{v}_n > 0 \text{ in } \Omega_n, \\
\bar{v}_n |_{\partial \Omega} = v
\end{cases}
\]

with \( g_1, g_2 \) defined in assumption (2.1) and \( h_1, h_2 : \Omega \times \mathbb{R} \to \mathbb{R}_+ \) the following cut-off functions:

\[
h_1(x, z) \overset{\text{def}}{=} \begin{cases}
g_1(x)u & \text{if } z \geq \bar{u}(x), \\
g_1(x)z & \text{if } z \in [0, \bar{u}(x)], \\
0 & \text{if } z \leq 0,
\end{cases}
\]

\[
h_2(x, z) \overset{\text{def}}{=} \begin{cases}
g_2(x)v & \text{if } z \geq \bar{v}(x), \\
g_2(x)z & \text{if } z \in [0, \bar{v}(x)], \\
0 & \text{if } z \leq 0.
\end{cases}
\]

From now on and without loss of generality, we will only study problem \((P^n_1)\) and conclude in the same way for problem \((P^n_2)\).
In problem (P_n^1), h_1 is a Carathéodory function on \( \Omega \times \mathbb{R} \). Thus, for a.e. \((x, s) \in \Omega \times \mathbb{R}\), setting
\[
H_1(x, s) \overset{\text{def}}{=} \int_0^s h_1(x, z) \, dz,
\]
we consider the following functional: \( \forall w \in W_0^{1,p}(\Omega_n) \),
\[
E_n^1(w) \overset{\text{def}}{=} \frac{1}{p} \left( \int_{\Omega_n} |\nabla (w + u)|^p \, dx + \int_{\Omega_n} H_1(x, w + u) \, dx \right.
\]
\[
+ \int_{\Omega_n} [f_1(x, u, v) - g_1(x)u] (w + u) \, dx.
\]
Then, \( E_n^1 \) is well defined in \( W_0^{1,p}(\Omega_n) \) and, taking into account that
\[
f_1(x, u, v) - g_1(x)u \in L^\infty(\Omega_n),
\]
from Hölder’s and Poincaré’s inequalities, we get: for any \( w \in W_0^{1,p}(\Omega_n) \),
\[
E_n^1(w) \geq \frac{1}{p} \left( \|\nabla w\|_{L^p(\Omega_n)} - \|\nabla u\|_{L^p(\Omega_n)} \right)^p
\]
\[
- \|f_1(x, u, v) + g_1(x)u\|_{L^{p'}(\Omega_n)} \left( (\lambda_{1,p})^{-\frac{1}{p}} \|\nabla w\|_{L^p(\Omega_n)} + \|u\|_{L^p(\Omega_n)} \right)
\] (2.3)
and \( E_n^1 \) is bounded from below in \( W_0^{1,p}(\Omega_n) \). So, let us define
\[
I_n^1 \overset{\text{def}}{=} \inf_{w \in W_0^{1,p} (\Omega_n)} E_n^1(w)
\] (2.4)
and let \( (w_k)_{k \in \mathbb{N}} \subset W_0^{1,p}(\Omega_n) \) be a minimizing sequence of \( E_n^1 \), i.e.
\[
\lim_{k \to +\infty} E_n^1(w_k) = I_n^1.
\]
Using (2.3) and Poincaré’s inequality, \( (w_k)_{k \in \mathbb{N}} \) is bounded in \( W_0^{1,p}(\Omega_n) \). Therefore, there exist a subsequence \( (w_{k(m)})_{m \in \mathbb{N}} \) and an element \( u_n^* \in W_0^{1,p}(\Omega_n) \) such that \( w_{k(m)} \xrightarrow{m \to +\infty} u_n^* \) weakly in \( W_0^{1,p}(\Omega_n) \) and a.e. in \( \Omega_n \). In particular,
\[
\liminf_{m \to +\infty} \|w_{k(m)}\|_{W_0^{1,p}(\Omega_n)} \geq \|u_n^*\|_{W_0^{1,p}(\Omega_n)}
\]
and using Fatou’s lemma,
\[
\liminf_{m \to +\infty} \int_{\Omega_n} H_1(x, w_{k(m)}) \, dx \geq \int_{\Omega_n} H_1(x, u_n^*) \, dx.
\]
Hence, $E_1^\alpha(u^\ast_n) = I_1^\alpha$ and $u^\ast_n$ is a solution to the following Euler-Lagrange equation:

\[
\int_{\Omega_n} |\nabla (u^\ast_n + u)|^{p-2} \nabla (u^\ast_n + u) \cdot \nabla w \, dx + \int_{\Omega_n} h_1(x, u^\ast_n + u) w \, dx = \int_{\Omega_n} (g_1(x)u - f_1(x, u, v)) \, w \, dx, \tag{2.5}
\]

Thus, $\tilde{u}_n \overset{\text{def}}{=} u^\ast_n + u \in W^{1, p}(\Omega_n)$ satisfies both $\tilde{u}_n |_{\partial \Omega_n} = u$, in the sense of the trace, and

\[
\Delta_p \tilde{u}_n - h_1(x, \tilde{u}_n) = f_1(x, u, v) - g_1(x)u \quad \text{in } D'(\Omega_n). \tag{2.6}
\]

Furthermore, since

\[
f_1(x, u, v) - g_1(x)u + h_1(x, \tilde{u}_n) \in L^\infty(\Omega_n),
\]

it follows from [28, Theorem 1] (see also [8, 34]) that $\tilde{u}_n \in C^{1, \beta'_n}(\overline{\Omega_n})$, for some $0 < \beta'_n < 1$, only depending on $p$, $N$, $\alpha$, $\|g_1\|_{L^\infty(\Omega_n)}$, $\|\overline{u}\|_{\partial(\overline{\Omega_n})}$ and $\|f_1\|_{L^\infty(\Omega_n \times [m_n, M_n])}$, with

\[
m_n \overset{\text{def}}{=} \min_{x \in \Omega_n} u(x) \quad \text{and} \quad M_n \overset{\text{def}}{=} \max_{x \in \Omega_n} \overline{u}(x).
\]

Now, it remains to prove that $\tilde{u}_n \in [u, \overline{u}]$. Then, combining (2.5) with (1.7), we get: $\forall w \in W_0^{1, p}(\Omega_n)^+ \overset{\text{def}}{=} \{ w \in W_0^{1, p}(\Omega_n), \; w \geq 0 \; \text{a.e in } \Omega_n \}$,

\[
\int_{\Omega_n} \left( |\nabla \tilde{u}_n|^{p-2} \nabla \tilde{u}_n - |\nabla u|^{p-2} \nabla u \right) \cdot \nabla w \, dx + \int_{\Omega_n} [h_1(x, \tilde{u}_n) - h_1(x, u)] \, w \, dx \geq \int_{\Omega_n} \left[ (g_1(x)u - f_1(x, u, v)) - (g_1(x)\overline{u} - f_1(x, u, v)) \right] \, w \, dx.
\]

By assumption $(H_1)$, applying this inequality with $w = (\tilde{u}_n - \overline{u})^- \in W_0^{1, p}(\Omega_n)^+$, we get (by monotonicity of the $p$-Laplace operator) that $\tilde{u}_n \geq \overline{u}$ in $\Omega_n$ and, on the way, that $\tilde{u}_n$ is a weak solution to $(P_1^n)$. Similarly, combining (2.5) with (1.9), we also get $\tilde{u}_n \leq \overline{u}$ in $\Omega_n$. Then, (2.6) becomes

\[
\Delta_p \tilde{u}_n - g_1(x)\tilde{u}_n = f_1(x, u, v) - g_1(x)u \quad \text{in } D'(\Omega_n). \tag{2.7}
\]

Finally, if we suppose that $\tilde{w}_n \in C^{1, \beta'_n}(\overline{\Omega_n})$ is another weak solution to $(P_1^n)$, $w^\ast_n \overset{\text{def}}{=} \tilde{w}_n - \overline{u} \in W_0^{1, p}(\Omega)$ satisfies (2.5). Thus, applying again this comparison
principle, one can show that \( \tilde{w}_n = \tilde{u}_n \) in \( \Omega_n \). And then, \((P_1^n)\) has a unique weak solution.

Let us denote
\[
C_n \overset{\text{def}}{=} \{ (u, v) \in C(\overline{\Omega}_n) \times C(\overline{\Omega}_n), \quad u \leq u \leq \tilde{u} \quad \text{and} \quad v \leq v \leq \tilde{v} \quad \text{in} \quad \Omega_n \},
\]
which is a convex set, closed for the usual uniform norm in \( C(\overline{\Omega}_n) \times C(\overline{\Omega}_n) \). Then, remarking that \( C_n \) is nothing more than the set of the restrictions to \( \overline{\Omega}_n \) of the functions of \( C \), we can compose an order reversing mapping:
\[
T^n : C_n \longrightarrow C_n \quad (u, v) \longmapsto (T^1_n(u, v), T^2_n(u, v)) \overset{\text{def}}{=} (\tilde{u}_n, \tilde{v}_n),
\]
where \( \tilde{u}_n \) (resp. \( \tilde{v}_n \)) is the unique solution to problem \((P_1^n)\) (resp. \((P_2^n)\)) we just study. With this notations, it is now clear that any fixed-point of the mapping \( T^n \) is a weak solution to \((P^n)\) and vice-versa. Precisely, we aim to apply here the Schauder fixed-point theorem. Hence, it just remains to prove that \( T^n \) is a continuous mapping and that \( T^n(C_n) \) is relatively compact subspace of \( C_n \).

- **(Relative compactness of \( T^n(C_n) \))**

  It directly follows from the uniform bound we get in the Hölder’s space \( C^{1,\beta'_n}(\overline{\Omega}_n) \). Precisely, it suffices to remark that since \((u, v) \in C\), in (2.7) the term \( f_1(x, u, v) - g_1(x)(u - \tilde{u}_n) \) is bounded in \( L^\infty(\Omega_n) \), uniformly in \( \Omega_n \times C_n \). Therefore, from [28, Theorem 1], there exists a constant \( C > 0 \) only depending on \( p, N, \Omega_n, \alpha, \| g_1 \|_{L^\infty(\Omega_n)}, \| \tilde{u} \|_{C(\overline{\Omega}_n)} \) and \( \| f_1 \|_{L^\infty(\Omega_n \times [m, M])} \) such that

\[
\forall (u, v) \in C_n, \quad \| T^1_n(u, v) \|_{C^{1,\beta'_n}(\overline{\Omega}_n)} \leq C. \tag{2.8}
\]

Similarly, we get the same kind of uniform \( C^{1,\beta'_n} \) estimate for \( T^2_n \) on \( C_n \) and we finally get the sufficient equicontinuity in \( C(\overline{\Omega}_n) \times C(\overline{\Omega}_n) \) to apply the Ascoli-Arzelá theorem.

- **(Continuity of \( T^n \))**

Let \((u^k, v^k)_{k \in \mathbb{N}} \subset C_n\) be a sequence converging to an element \((u, v) \in C_n\) as \( k \to +\infty \) for the usual uniform norm in \( C(\overline{\Omega}_n) \times C(\overline{\Omega}_n) \). For any \( k \in \mathbb{N} \), let us denote \( T^m_1(u^k, v^k) = \tilde{u}^k_n \in C^{1,\beta'_n}(\overline{\Omega}_n) \), the unique solution to \((P^n_1)\) associated to \((u^k, v^k)\). It follows from (2.8) and the well known compact embedding \( C^{1,\beta'_n}(\overline{\Omega}_n) \hookrightarrow W^{1,p}(\Omega_n) \cap C(\overline{\Omega}_n) \) that there exist a subsequence \((\tilde{u}^{k(m)}_n)_{m \in \mathbb{N}}\) and \( \phi_n \in W^{1,p}(\Omega_n) \cap C(\overline{\Omega}_n) \) such that

\[
\tilde{u}^{k(m)}_n \xrightarrow{m \to +\infty} \phi_n \quad \text{in} \quad W^{1,p}(\Omega_n) \cap C(\overline{\Omega}_n). \tag{2.9}
\]
Then, passing to the limit as \( m \to +\infty \), \( \phi_n \) satisfies together \( \phi_n|_{\partial \Omega_n} = u \), \( u \leq \phi_n \leq \overline{u} \) in \( \Omega_n \) and

\[
\Delta \phi_n - g_1(x)\phi_n = f_1(x, u, v) - g_1(x)u \quad \text{in} \quad \mathcal{D}'(\Omega_n).
\]

From the uniqueness of the solution to \( (P^n) \), it follows that \( \phi_n = T^n_1(u, v) \). Finally, using a standard argument relying on the uniqueness of the solution to \( (P^n) \), we prove that all the sequence \( (\tilde{u}^k_n)_{k \in \mathbb{N}} \) converges to \( T^n_1(u, v) \). And we complete the proof of the continuity of \( T^n \) arguing in the same way for problem \( (P^n_2) \).

Then, from the Schauder fixed-point theorem, we get a fixed-point \( (u_n, v_n) \in C_n \) of the mapping \( T^n \) which is a weak solution to \( (P^n) \). Now, for any \( k \geq n \), we define the functions \( u^k_n, v^k_n \) on the whole domain \( \Omega \) as follows:

\[
u^k_n \equiv \mathbb{I}_{\Omega_n} \cdot u_k + \mathbb{I}_{\Omega \setminus \Omega_n} \cdot u \quad \text{in} \quad \Omega,
\]

\[
u^k_n \equiv \mathbb{I}_{\Omega_n} \cdot v_k + \mathbb{I}_{\Omega \setminus \Omega_n} \cdot v \quad \text{in} \quad \Omega.
\]

Then for \( n = 1 \), from above estimate (2.8), the sequence \( (u^k_1)_{k \geq 1} \) is bounded in \( C^{1,\beta'}(\overline{\Omega}_1) \). Therefore, since \( C^{1,\beta'}(\overline{\Omega}_1) \hookrightarrow C^1(\overline{\Omega}_1) \), there exist a subsequence \( (u^k_1(m))_{m \in \mathbb{N}} \) and \( u^*_1 \in C^1(\overline{\Omega}_1) \) such that

\[
u^k_1(m) \xrightarrow{m \to +\infty} u^*_1 \quad \text{in} \quad C^1(\overline{\Omega}_1). \tag{2.10}
\]

In the same way, by iteration we construct for any \( n \geq 2 \), \( (u^k_n(m))_{m \in \mathbb{N}} \) a subsequence of \( (u^k_{n-1}(m))_{m \in \mathbb{N}} \) and \( u^*_n \in C^1(\overline{\Omega}_n) \) such that

\[
u^k_n(m) \xrightarrow{m \to +\infty} u^*_n \quad \text{in} \quad C^1(\overline{\Omega}_n). \tag{2.11}
\]

However, by definition we have for any \( n' \geq n \) and any \( m \in \mathbb{N} \),

\[
u^k_{n'}(m) = u^k_{n'}(m) \quad \text{in} \quad \overline{\Omega}_n.
\]

Hence, passing to the limit as \( m \to +\infty \), it follows that \( u^*_n = u^*_n \) in \( \overline{\Omega}_n \). Taking into account of this remark, we can construct a function \( u^* \in C^{1,\beta}(\Omega) \), such that \( u \leq u^* \leq \overline{u} \) in \( \Omega \), as follows: \( \forall x \in \Omega \),

\[
u^*(x) \equiv u^*_i(x), \quad i \geq n_0(x),
\]

with \( n_0(x) \in \mathbb{N}^* \) the unique integer satisfying

\[
\frac{D}{n_0(x) + 1} \leq d(x, \partial \Omega) < \frac{D}{n_0(x)}.
\]
Let $\Omega' \subset \subset \Omega$ and $n_{\Omega'} \in \mathbb{N}^*$ the unique integer such that
\[ \Omega_{n_{\Omega'}} \subset \Omega' \subset \Omega_{n_{\Omega'} + 1}. \]

Therefore, by a Cantor’s diagonal argument, the sequence $(u_{km(m)})_{m \geq n_{\Omega'}}$ converges to $u^*$ in $C^1(\Omega')$ as $m \to +\infty$. Then, we construct in the same way a function $v^* \in C^1_{loc}(\Omega)$ such that $u \leq v^* \leq v$ in $\Omega$, and passing to the limit as $m \to +\infty$ in (2.7) and in the similar equation satisfied by $\tilde{v}_n$, $(u^*, v^*)$ is a large solution to (P). Finally, the $C^1, \beta_{loc}(\Omega) \times C^1, \beta_{loc}(\Omega)$ regularity of $(u^*, v^*)$ follows from the well-known interior regularity result [8, Theorem 2] (see also [34, Theorem 2]).

\section{Applications}

In this section we focus on the following quasilinear elliptic and singular system:
\[
(\text{P}) \begin{cases}
\Delta_p u = K_1(x) u^{a_1} v^{b_1} \quad \text{in } \Omega; & u|_{\partial \Omega} = +\infty, \quad u > 0 \quad \text{in } \Omega, \\
\Delta_q v = K_2(x) v^{a_2} u^{b_2} \quad \text{in } \Omega; & v|_{\partial \Omega} = +\infty, \quad v > 0 \quad \text{in } \Omega.
\end{cases}
\]

In this problem,

(i) Exponents $a_1 > p - 1, a_2 > q - 1$ and $b_1, b_2 \neq 0$ satisfy the super-homogeneity condition:
\[
(a_1 - p + 1)(a_2 - q + 1) - b_1 b_2 > 0; \tag{3.1}
\]
which is equivalent to the existence of a positive constant $\sigma > 0$ such that
\[
(a_1 - p + 1) - \sigma|b_1| > 0 \quad \text{and} \quad \sigma(a_2 - q + 1) - |b_2| > 0. \tag{3.2}
\]

(ii) $K_1, K_2$ are two positive functions of $C(\Omega)$ satisfying
\[
K_1(x) \approx d(x)^{-\lambda_1} L_1(d(x)) \quad \text{and} \quad K_2(x) \approx d(x)^{-\lambda_2} L_2(d(x)) \quad \text{in } \Omega, \tag{3.3}
\]
where $\lambda_1 < p, \lambda_2 < q$ and for $i = 1, 2$, $L_i$ is a Karamata function, i.e. a lower perturbation in $C^2((0, \eta])$, with $\eta > D$, which can be written in the form:
\[
\forall t \in (0, \eta], \quad L_i(t) = \exp \left( \int_t^\eta \frac{z_i(s)}{s} ds \right), \tag{3.4}
\]
with $z_i \in C([0, \eta]) \cap C^1((0, \eta])$ satisfying $z_i(0) = 0$ and $\lim_{t \to 0^+} tz_i'(t) = 0$. We denote by $\mathcal{K}$ the Karamata class (see [24, 25]) of all functions $L_i$ satisfying the above definition.
Remark 3.1. We just mention here some useful properties of the Karamata functions we use next. The interested reader should find the proofs in [1, §2] and more details on the topic in [24, 25].

a. Let \( L_1, L_2 \in K \) and \( p_1, p_2 \in \mathbb{R} \). Then, \( L_1^{p_1} \cdot L_2^{p_2} \) also belongs to \( K \).

b. Let \( L \in K \). Then, it satisfies the following asymptotic behaviour:
\[
\forall \varepsilon > 0, \quad \lim_{t \to 0^+} t^{-\varepsilon} L_i(t) = +\infty \quad \text{and} \quad \lim_{t \to 0^+} t^\varepsilon L_i(t) = 0. \quad (3.5)
\]

c. A function \( L \) is in \( K \) if and only if \( L \) is a positive function of \( C^2([0, \eta]) \) satisfying
\[
\lim_{t \to 0^+} \frac{tL'(t)}{L(t)} = \lim_{t \to 0^+} \frac{t^2L''(t)}{L(t)} = 0. \quad (3.6)
\]

d. Let \( L \in K \) and \( \varepsilon > 0 \). Then, from (1.5) and the mean value theorem, the asymptotic behaviour of \( L(\varepsilon \varphi_1, r) \) is given by,
\[
L(\varepsilon \varphi_1, r(x)) \approx L(d(x)) \quad \text{in} \ \Omega. \quad (3.7)
\]

Example 3.2. Let \( m \in \mathbb{N}^*, C > 0 \) and \( \omega >> \eta \) large enough. Let us define
\[
\forall t \in (0, \eta], \quad L_i(t) = C \prod_{n=1}^m \left( \log_n \left( \frac{\omega}{t} \right) \right)^{\mu_n},
\]

where, \( \log_n \overset{\text{def}}{=} \log \circ \cdots \circ \log (n \text{ times}) \) and \( \mu_n \in \mathbb{R} \). Then, \( L_i \) belongs to \( K \).

We discuss here the existence of large solutions to problem \((P)\). For that, regarding Theorem 2.1, we take
\[
f_1(x, u, v) = K_1(x)u^{a_1}v^{b_1} \quad \text{and} \quad f_2(x, u, v) = K_2(x)v^{a_2}u^{b_2}
\]

and construct a suitable sub and supersolutions pair to \((P)\).

Preliminary results

Let \( 1 < r < +\infty \). Then, let \( \delta > r - 1 \) and \( K \in \mathcal{C}(\Omega) \), such that
\[
K(x) \approx d(x)^{-\lambda} L(d(x)) \quad \text{in} \ \Omega, \quad (3.8)
\]

with \( \lambda \leq r \) and \( L \in K \) a lower perturbation function satisfying (3.4). In view of constructing a suitable sub and supersolutions pair to \((P)\), we first introduce the following problem:
\[
(Q) \left\{ \begin{array}{l}
\Delta_r w = K(x)w^{\delta}, \quad w > 0 \quad \text{in} \ \Omega \\
w|_{\partial\Omega} = +\infty.
\end{array} \right.
\]
Proposition 3.3. Under the above hypotheses,

(i) Assume that $\lambda < r$. Then, problem (Q) has a positive solution $w \in C^1_{loc}(\Omega)$, for some $0 < \beta' < 1$, which satisfies

\[ w(x) \approx d(x)^{\frac{r-\lambda}{r-1-\delta}} L(d(x))^{\frac{1}{r-1-\delta}} \text{ in } \Omega. \tag{3.9} \]

(ii) Assume that $\lambda = r$ and assume in addition that $L$ satisfies the following integrability condition:

\[ \int_0^{\eta} t^{-1} L(t)^{\frac{1}{r-1}} \, dt < +\infty. \tag{3.10} \]

Then, problem (Q) has a positive solution $w \in C^1_{loc}(\Omega)$, for some $0 < \beta' < 1$, which satisfies

\[ w(x) \approx \left( \int_0^{d(x)} t^{-1} L(t)^{\frac{1}{r-1}} \, dt \right)^{\frac{r-1}{r-1-\delta}} \text{ in } \Omega. \tag{3.11} \]

Proof. See section A in Appendix.

Remark 3.4.

(i) This result extends [1, Theorem 3] to the $r$-Laplace operator.

(ii) This result may easily be generalized to quasilinear problems with (possible singular) weights $K(x)$ shaped as (3.8), where exponent $\lambda = \lambda(x)$ is a Hölder’s continuous function in $\Omega$. As a consequence, it also extends the existence result of [14, Theorem 1] to quasilinear problems with (possible singular) weights containing a Karamata perturbation.

(iii) We recall that the aim of the study of problem (Q) is to guarantee the existence of suitable sub and supersolutions pairs in order to apply Theorem 2.1 in the example of this section. For this reason, we deliberately occult here any question about the non-existence or the uniqueness of solution. Nevertheless, may notice that, due to the presence of a Karamata perturbation in the weighted term $K(x)$, the question of the uniqueness, particularly, seems not obvious and, to our knowledge, has not been considered yet in the previous papers mentioned in [14].
3.1 Main Results

**Theorem 3.5.** Assume that exponents \( a_1, a_2 \in \mathbb{R} \) and \( b_1, b_2 \neq 0 \) in problem (P) satisfy hypothesis (3.1). Set

\[
\alpha_1 = \frac{q - 1 - a_2}{(a_1 - p + 1)(a_2 - q + 1) - b_1 b_2}, \quad \alpha_2 = \frac{p - 1 - a_1}{(a_1 - p + 1)(a_2 - q + 1) - b_1 b_2},
\]

\[
\beta_1 = \frac{b_1}{(a_1 - p + 1)(a_2 - q + 1) - b_1 b_2}, \quad \beta_2 = \frac{b_2}{(a_1 - p + 1)(a_2 - q + 1) - b_1 b_2},
\]

\[
\gamma_1 = \frac{(p - \lambda_1)(q - 1 - a_2) + (q - \lambda_2)b_1}{(a_1 - p + 1)(a_2 - q + 1) - b_1 b_2}, \quad \gamma_2 = \frac{(q - \lambda_2)(p - 1 - a_1) + (p - \lambda_1)b_2}{(a_1 - p + 1)(a_2 - q + 1) - b_1 b_2}
\]

and assume that

\[
(p - \lambda_1)(q - 1 - a_2) + (q - \lambda_2)b_1 < 0, \quad (3.12)
\]

\[
(q - \lambda_2)(p - 1 - a_1) + (p - \lambda_1)b_2 < 0. \quad (3.13)
\]

Then, problem (P) possesses a large solution \((u^*, v^*) \in \mathcal{C}^{1,\beta'}_{\text{loc}}(\Omega) \times \mathcal{C}^{1,\beta}_{\text{loc}}(\Omega), \) for some \( 0 < \beta < 1, \) that satisfies the following estimates:

\[
u^*(x) \approx d(x)^{\gamma_1} L_1(d(x))^\alpha_1 L_2(d(x))^\beta_1 \quad \text{in } \Omega, \quad (3.14)
\]

\[
u^*(x) \approx d(x)^{\gamma_2} L_2(d(x))^\alpha_2 L_1(d(x))^\beta_2 \quad \text{in } \Omega. \quad (3.15)
\]

**Remark 3.6.**

(i) Assume that \( L_1(t)^\alpha L_2(t)^\beta \xrightarrow{t \to 0^+} +\infty. \) Then, conclusion of the above theorem still holds if

\[
(p - \lambda_1)(q - 1 - a_2) + (q - \lambda_2)b_1 = 0.
\]

And symmetrically, assume that \( L_2(t)^\alpha L_1(t)^\beta \xrightarrow{t \to 0^+} +\infty. \) Then, conclusion of the above theorem still holds if

\[
(q - \lambda_2)(p - 1 - a_1) + (p - \lambda_1)b_2 = 0.
\]

(ii) This theorem unifies the different existence results stated in [3, 13, 36].

(iii) For the same reason as in point (iii) of Remark 3.4, we do not investigate here the questions of non-existence or uniqueness of large solutions to (P). Nevertheless, concerning the non-existence, from the necessary and sufficient condition on the existence of a large solution given in [13, Theorem 1], conditions (3.12) and (3.13) seem to be sharp.
Proof. (of Theorem 3.5) We adapt here the method used in [17, 21] to construct a sub and supersolutions pair to (P). Thanks to Proposition 3.3, we apply Theorem 2.1 with a suitable choice of sub and supersolutions pair \((u, v), (\overline{u}, \overline{v})\) in \(\mathcal{C}_{\text{loc}}^{1,\beta'}(\Omega) \times \mathcal{C}_{\text{loc}}^{1,\beta}(\Omega)\), for some \(0 < \beta' < 1\), in the following form:

\[
\overline{u} = m\psi_1 \quad \text{and} \quad \overline{u} = m^{-1}\psi_1 \quad \text{in} \ \Omega, \quad (3.16)
\]

\[
\overline{v} = m^\sigma \psi_2 \quad \text{and} \quad \overline{v} = m^{-\sigma} \psi_2 \quad \text{in} \ \Omega, \quad (3.17)
\]

where \(\sigma > 0\) is given in (3.2), \(0 < m < 1\) is an appropriate constant small enough and \(\psi_1, \psi_2 \in \mathcal{C}_{\text{loc}}^{1,\beta'}(\Omega)\) are given by Proposition 3.3 as respective solutions to problems

\[
(Q_1) \begin{cases}
\Delta_p w = d(x)^{-\lambda_1} L_1(d(x)) w^{\delta_1}, & w > 0 \quad \text{in} \ \Omega, \\
|w|_{\partial \Omega} = +\infty
\end{cases}
\]

and

\[
(Q_2) \begin{cases}
\Delta_q w = d(x)^{-\lambda_2} L_2(d(x)) w^{\delta_2}, & w > 0 \quad \text{in} \ \Omega, \\
|w|_{\partial \Omega} = +\infty
\end{cases}
\]

satisfying some cone conditions we specify below. We choose suitable perturbations \(L_1, L_2 \in \mathcal{K}\) and suitable values of exponents \(\delta_1, \delta_2 > 1\) in order to satisfy

\[
\Delta_p \psi_1 \approx K_1(x) \psi_1^{a_1} \psi_2^{b_1} \quad \text{and} \quad \Delta_q \psi_2 \approx K_2(x) \psi_2^{a_2} \psi_1^{b_2} \quad \text{in} \ \Omega; \quad (3.18)
\]

which provide us inequalities (1.7) to (1.10) in order to apply Theorem 2.1. Here, we look for large solutions \((u^*, v^*) \in \mathcal{C}_{\text{loc}}^{1,\beta}(\Omega) \times \mathcal{C}_{\text{loc}}^{1,\beta}(\Omega)\), for some \(0 < \beta < 1\), to (P) by making the "Ansatz" that

\[
u^*(x) \approx d(x)^{\gamma_1} L_1(d(x))^{\alpha_1} L_2(d(x))^{\beta_1} \quad \text{in} \ \Omega,
\]

\[
u^*(x) \approx d(x)^{\gamma_2} L_2(d(x))^{\alpha_2} L_1(d(x))^{\beta_2} \quad \text{in} \ \Omega,
\]

for some \(\gamma_1, \gamma_2 < 0\) (or \(\gamma_1, \gamma_2 = 0\) under assumption of Remark 3.6) and \(\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}\). For that, we take in problems (Q_1) and (Q_2):

\[
L_1 = L_1^{\nu_1} \cdot L_2^{\mu_1} \quad \text{and} \quad L_2 = L_2^{\nu_2} \cdot L_1^{\mu_2} \quad \text{in} \ \Omega,
\]

where \(\nu_1, \nu_2, \mu_1, \mu_2 \in \mathbb{R}\) are suitable exponents we fix later. By Proposition 3.3, \(\psi_1, \psi_2 \in \mathcal{C}_{\text{loc}}^{1,\beta'}(\Omega)\), for some \(0 < \beta' < 1\) and satisfy

\[
\psi_1(x) \approx d(x)^{\frac{p-\lambda_1}{p-1-\delta_1}} L_1(d(x))^{\frac{\nu_1}{p-1-\delta_1}} L_2(d(x))^{\frac{\mu_1}{p-1-\delta_1}} \quad \text{in} \ \Omega, \quad (3.19)
\]
\[ \psi_2(x) \approx d(x)^{\frac{q - \lambda_2}{q - 1 - \delta_2}} L_2(d(x))^{\frac{\nu_2}{q - 1 - \delta_2}} L_1(d(x))^{\frac{\mu_2}{q - 1 - \delta_2}} \quad \text{in } \Omega. \tag{3.20} \]

In view of satisfying estimates given in (3.18), the comparison of the term \( \Delta \psi_1 \) with \( K_1(x) \psi_1^{a_1} \psi_2^{b_1} \) on one side, and the term \( \Delta \psi_2 \) with \( K_2(x) \psi_2^{a_2} \psi_1^{b_2} \) on the other side, imposes the exponents \( \nu_1, \nu_2, \mu_1, \mu_2 \) and \( \delta_1, \delta_2 \) to satisfy the following system:

\[
\begin{align*}
\delta_1 \frac{p - \lambda_1}{p - 1 - \delta_1} &= a_1 \frac{p - \lambda_1}{p - 1 - \delta_1} + b_1 \frac{q - \lambda_2}{q - 1 - \delta_2}, \\
\delta_2 \frac{q - \lambda_2}{q - 1 - \delta_2} &= a_2 \frac{q - \lambda_2}{q - 1 - \delta_2} + b_2 \frac{p - \lambda_1}{p - 1 - \delta_1}, \\
\nu_1 \frac{p - 1}{p - 1 - \delta_1} &= 1 + a_1 \frac{\lambda_1}{p - 1 - \delta_1} + b_1 \frac{\mu_2}{q - 1 - \delta_2}, \\
\nu_2 \frac{q - 1}{q - 1 - \delta_2} &= 1 + a_2 \frac{\lambda_2}{q - 1 - \delta_2} + b_2 \frac{\mu_1}{p - 1 - \delta_1}, \\
\mu_1 \frac{p - 1}{p - 1 - \delta_1} &= a_1 \frac{\mu_1}{p - 1 - \delta_1} + b_1 \frac{\nu_2}{q - 1 - \delta_2}, \\
\mu_2 \frac{q - 1}{q - 1 - \delta_2} &= a_2 \frac{\mu_2}{q - 1 - \delta_2} + b_2 \frac{\nu_1}{p - 1 - \delta_1}.
\end{align*}
\]

Then, we get

\[
\begin{align*}
\gamma_1 &= \frac{p - \lambda_1}{p - 1 - \delta_1} = \frac{(p - \lambda_1)(q - 1 - a_2) + (q - \lambda_2)b_1}{(a_1 - p + 1)(a_2 - q + 1) - b_1 b_2}, \tag{3.21} \\
\gamma_2 &= \frac{q - \lambda_2}{q - 1 - \delta_2} = \frac{(q - \lambda_2)(p - 1 - a_1) + (p - \lambda_1)b_2}{(a_1 - p + 1)(a_2 - q + 1) - b_1 b_2}, \tag{3.22} \\
\alpha_1 &= \frac{\nu_1}{p - 1 - \delta_1} = \frac{q - 1 - a_2}{(a_1 - p + 1)(a_2 - q + 1) - b_1 b_2}, \tag{3.23} \\
\alpha_2 &= \frac{\nu_2}{q - 1 - \delta_2} = \frac{p - 1 - a_1}{(a_1 - p + 1)(a_2 - q + 1) - b_1 b_2}, \tag{3.24} \\
\beta_1 &= \frac{\mu_1}{p - 1 - \delta_1} = \frac{b_1}{(a_1 - p + 1)(a_2 - q + 1) - b_1 b_2}, \tag{3.25} \\
\beta_2 &= \frac{\mu_2}{q - 1 - \delta_2} = \frac{b_1}{(a_1 - p + 1)(a_2 - q + 1) - b_1 b_2}. \tag{3.26}
\end{align*}
\]

Then, under assumption (3.1), to get \( \gamma_1, \gamma_2 < 0 \), we have to impose conditions (3.12) and (3.13) on exponents \( a_1, a_2, b_1, b_2 \) and \( \lambda_1, \lambda_2 \).
Now, we would like to find a suitable constant $0 < m < 1$ in order to satisfy inequalities (1.7) to (1.10) for $(u, v)$, $(\bar{u}, \bar{v}) \in C_{\text{loc}}^{1, \beta}(\Omega) \times C_{\text{loc}}^{1, \beta}(\Omega)$ defined in (3.16) and (3.17). Then, let $(u, v) \in [u, \bar{u}] \times [v, \bar{v}]$.

(i) Concerning the subsolutions inequalities (1.7) and (1.8). On one hand we have estimates (3.19) and (3.20)

$$\Delta_p u \geq m C_1 L_1 (d(x))^{\mu_1 + \delta_1 \alpha_1} L_2 (d(x))^{\mu_1 + \delta_1 \beta_1} d(x)^{\delta_1 \gamma_1 - \lambda_1} \text{ in } \Omega,$$

$$\Delta_q v \geq m C_1' L_2 (d(x))^{\nu_2 + \delta_2 \alpha_2} L_1 (d(x))^{\nu_2 + \delta_2 \beta_2} d(x)^{\delta_2 \gamma_2 - \lambda_2} \text{ in } \Omega.$$

And on the other hand,

$$K_1(x) \bar{u}^{a_1} \bar{v}^{b_1} \leq C_2 m^{-\sigma |b_1|} \Lambda_1 (d(x)) d(x)^{a_1 \gamma_1 + b_1 \gamma_1 - \lambda_1} \text{ in } \Omega,$$

$$K_2(x) \bar{v}^{a_2} \bar{u}^{b_2} \leq C'_2 m^{\sigma a_2 - |b_2|} \Lambda_2 (d(x)) d(x)^{a_2 \gamma_2 + b_2 \gamma_2 - \lambda_2} \text{ in } \Omega,$$

with $\Lambda_1 = L_1^{1-a_1 \alpha_1 + b_1 \beta_2} \cdot L_2^{a_1 \alpha_1 + b_1 \beta_2}$ and $\Lambda_2 = L_2^{1-a_2 \alpha_2 + b_2 \beta_1} \cdot L_1^{a_2 \beta_2 + b_2 \alpha_1}$.

(ii) Concerning the supersolutions inequalities (1.9) and (1.10), we get similarly on one hand

$$\Delta_p \bar{u} \leq m^{-1} C_3 L_1 (d(x))^{\nu_1 + \delta_1 \alpha_1} L_2 (d(x))^{\mu_1 + \delta_1 \beta_1} d(x)^{\delta_1 \gamma_1 - \lambda_1} \text{ in } \Omega,$$

$$\Delta_q \bar{v} \leq m^{-\sigma} C'_3 L_2 (d(x))^{\nu_2 + \delta_2 \alpha_2} L_1 (d(x))^{\mu_2 + \delta_2 \beta_2} d(x)^{\delta_2 \gamma_2 - \lambda_2} \text{ in } \Omega.$$

And on the other hand,

$$K_1(x) \bar{v}^{a_1} \bar{u}^{b_1} \geq C_4 m^{-a_1 + \sigma |b_1|} \Lambda_1 (d(x)) d(x)^{a_1 \gamma_1 + b_1 \gamma_1 - \lambda_1} \text{ in } \Omega,$$

$$K_2(x) \bar{u}^{a_2} \bar{v}^{b_2} \geq C'_4 m^{-\sigma a_2 + |b_2|} \Lambda_2 (d(x)) d(x)^{a_2 \gamma_2 + b_2 \gamma_2 - \lambda_2} \text{ in } \Omega.$$

Then from assumption (3.2) and thanks to equalities (3.21) to (3.26), $(u, v)$, $(\bar{u}, \bar{v})$ is a sub and supersolutions pair shaped has (3.16) and (3.17). Unfortunately, this approach does not allow us to investigate the critical case $\lambda_1 = p$ or $\lambda_2 = q$. Indeed, in this case, we cannot construct a suitable sub and supersolutions pair shaped has (3.16) and (3.17).
4 Other examples of applications

In this section, we give two extra examples of application of Theorem 2.1 for other Gierer-Meinhardt systems also arising in population dynamics, modelling more complex behaviours between two biological species than in the previous section. From now, for sake of clarity we do not consider right-hand sides containing singular weights $K_i(x)$ as in (3.3) anymore. Nevertheless, all the results stated below can be easily generalised to systems with singular weights in this form.

4.1 Second example

We consider now the following elliptic system:

\[
\begin{align*}
\Delta_p u &= u^{a_1}v^{b_1} - u^{a_1}v^{b_1} \quad \text{in } \Omega; \quad u|_{\partial\Omega} = +\infty, \quad u > 0 \quad \text{in } \Omega, \\
\Delta_q v &= v^{a_2}u^{b_2} - v^{a_2}u^{b_2} \quad \text{in } \Omega; \quad v|_{\partial\Omega} = +\infty, \quad v > 0 \quad \text{in } \Omega,
\end{align*}
\]

where the above exponents satisfy

\[
(a_1 - p + 1) - \sigma|b_1| > 0 \quad \text{and} \quad (a_1 - \alpha_1) - \sigma(|b_1| + |\beta_1|) > 0,
\]

\[
\sigma(a_2 - q + 1) - |b_2| > 0 \quad \text{and} \quad \sigma(a_2 - \alpha_2) - (|b_2| + |\beta_2|) > 0,
\]

for some constant $\sigma > 0$. Then, we have the following result:

**Theorem 4.1.** Set

\[
\gamma_1 = \frac{p(q - 1 - a_2) + qb_1}{(a_1 - p + 1)(a_2 - q + 1) - b_1b_2},
\]

\[
\gamma_2 = \frac{q(p - 1 - a_1) + pb_2}{(a_1 - p + 1)(a_2 - q + 1) - b_1b_2},
\]

and assume that

\[
p(q - 1 - a_2) + qb_1 < 0 \quad \text{and} \quad (\alpha_1 - a_1)\gamma_1 + (\beta_1 - b_1)\gamma_2 > 0,
\]

\[
q(p - 1 - a_1) + pb_2 < 0 \quad \text{and} \quad (\alpha_2 - a_2)\gamma_2 + (\beta_2 - b_2)\gamma_1 > 0.
\]

Then, problem (P) possesses a large solution $(u^*, v^*) \in C^1_{\text{loc}}(\Omega) \times C^1_{\text{loc}}(\Omega)$, for some $0 < \beta < 1$, that satisfies the following estimates:

\[
u^*(x) \sim d(x)^{\gamma_1} \quad \text{and} \quad v^*(x) \sim d(x)^{\gamma_2} \quad \text{in } \Omega.
\]
Proof. We apply Theorem 2.1 with

\[ u = m \psi_1, \quad \overline{u} = m^{-1} \psi_1 \quad \text{and} \quad v = m^\sigma \psi_2, \quad \overline{v} = m^{-\sigma} \psi_2 \quad \text{in} \ \Omega, \quad (4.8) \]

where \( \sigma > 0 \) is the constant given in (4.1) and (4.2), \( 0 < m < 1 \) is a positive constant small enough and \( \psi_1, \psi_2 \in C_{\text{loc}}^{1,\beta'}(\Omega) \), for some \( 0 < \beta' < 1 \), are given by Proposition 3.3 as respective solutions to problems

\[
\begin{equation}
(Q_1) \quad \left\{ \begin{array}{l}
\Delta_p w = w^{\delta_1}, \quad w > 0 \quad \text{in} \ \Omega, \\
|w|_{\partial \Omega} = +\infty
\end{array} \right.
\end{equation}
\]

and

\[
\begin{equation}
(Q_2) \quad \left\{ \begin{array}{l}
\Delta_q w = w^{\delta_2}, \quad w > 0 \quad \text{in} \ \Omega, \\
|w|_{\partial \Omega} = +\infty
\end{array} \right.
\end{equation}
\]

with suitable exponents \( \delta_1 > 1 - \frac{1}{p} \), \( \delta_2 > 1 - \frac{1}{q} \) such that \( \psi_1 \) and \( \psi_2 \) satisfies

\[
\Delta_p \psi_1 \approx \psi_1^{a_1} \psi_2^{b_1} \quad \text{and} \quad \Delta_q \psi_2 \approx \psi_2^{a_2} \psi_1^{b_2} \quad \text{in} \ \Omega. \quad (4.9)
\]

From Proposition 3.3,

\[
\psi_1(x) \approx d(x)^{\gamma_1} \quad \text{and} \quad \psi_2(x) \approx d(x)^{\gamma_2} \quad \text{in} \ \Omega, \quad (4.10)
\]

with \( \gamma_1 = \frac{p}{p-1-\delta_1} \) and \( \gamma_2 = \frac{q}{q-1-\delta_2} \). Therefore, similarly to the previous proof, the above condition (4.9) imposes exponents \( a_1, a_2, b_1, b_2 \) and \( \delta_1, \delta_2 \) to satisfy:

\[
\left\{ \begin{array}{l}
\frac{\delta_1 p}{p - 1 - \delta_1} = \frac{a_1 p}{p - 1 - \delta_1} + \frac{b_1 q}{q - 1 - \delta_2}, \\
\frac{\delta_2 q}{q - 1 - \delta_2} = \frac{a_2 q}{q - 1 - \delta_2} + \frac{b_2 p}{p - 1 - \delta_2}.
\end{array} \right.
\]

That is to say, \( \gamma_1 \) and \( \gamma_2 \) are given by (4.3) and (4.4) and must be negative.

Now, let \((u, v) \in [u, \overline{u}] \times [v, \overline{v}].\)

(i) Concerning the subsolutions inequalities (1.7) and (1.8). On one hand we have using estimates (4.10),

\[
\Delta_p u \geq m C_1 d(x)^{\delta_1 \gamma_1} \quad \text{and} \quad \Delta_q v \geq m^\sigma C'_1 d(x)^{\delta_2 \gamma_2} \quad \text{in} \ \Omega.
\]

And in the other hand,

\[
\begin{align*}
 u^{a_1} v^{b_1} - u^{a_1} v^{b_1} &\leq m^{a_1 - \sigma |b_1|} C_2 d(x)^{a_1 \gamma_1 + b_1 \gamma_2} \quad \text{in} \ \Omega, \\
v^{a_2} u^{b_2} - v^{a_2} u^{b_2} &\leq m^{a_2 - \sigma |b_2|} C'_2 d(x)^{a_2 \gamma_2 + b_2 \gamma_1} \quad \text{in} \ \Omega.
\end{align*}
\]
(ii) Concerning the supersolutions inequalities (1.9) and (1.10), we get similarly on one hand

\[ \Delta_p \overline{u} \leq m^{-1} C_3 d(x) \delta_1^{\gamma_1} \quad \text{and} \quad \Delta_q \overline{v} \geq m^{-\sigma} C_3 d(x) \delta_2^{\gamma_2} \quad \text{in} \ \Omega. \]

And on the other hand, from the second inequalities in assumptions (4.1) and (4.5), we get for \( m \) small enough

\[
\overline{v}^a_1 v^b_1 - \overline{v}^a_1 v^b_1 \geq m^{\sigma |b_1| - a_1} \psi_1^{a_1} \psi_2^{b_1} \left[ 1 - m^{(a_1 - a_1) - \sigma(\|\beta_1\| + |b_1|)} \psi_1^{a_1 - a_1} \psi_2^{b_1 - b_1} \right] \\
\geq m^{\sigma |b_1| - a_1} C_4 d(x)^{a_1 \gamma_1 + b_1 \gamma_2} \quad \text{in} \ \Omega,
\]

and from the second inequalities in assumptions (4.2) and (4.6),

\[
\overline{v}^a_1 v^b_2 - \overline{v}^a_1 v^b_2 \geq m^{\sigma |b_2| - \sigma \alpha_2} \psi_2^{a_2} \psi_2^{b_2} \left[ 1 - m^{(\sigma \alpha_2 - \alpha_2) - (|\beta_2| + |b_2|)} \psi_2^{a_2 - a_2} \psi_1^{b_2 - b_2} \right] \\
\geq m^{\sigma |b_2| - \sigma \alpha_2} C_4 d(x)^{a_2 \gamma_2 + b_2 \gamma_1} \quad \text{in} \ \Omega,
\]

Then, from the first inequalities in (4.1) and (4.2) and thanks to the choice of \( \gamma_1, \gamma_2, \ (u, v), (\overline{u}, \overline{v}) \) is a sub and supersolutions pair to problem (P), for \( m \) small enough. And we conclude in the same way as in the proof of Theorem 3.5.

\[ \square \]

### 4.2 Third example

Finally, we consider for instance the following elliptic system:

\[
\begin{aligned}
(P) \quad \begin{cases}
\Delta_p u = u^{a_1} v^{b_1} + \mu_1 u^{\alpha_1} + \nu_1 v^{\beta_1} & \text{in} \ \Omega; \quad u|_{\partial \Omega} = +\infty, \quad u > 0 \quad \text{in} \ \Omega, \\
\Delta_q v = v^{a_2} u^{b_2} + \mu_2 v^{\alpha_2} + \nu_2 v^{\beta_2} & \text{in} \ \Omega; \quad v|_{\partial \Omega} = +\infty, \quad v > 0 \quad \text{in} \ \Omega,
\end{cases}
\end{aligned}
\]

where \( \mu_1, \mu_2 \in \mathbb{R}, \nu_1, \nu_2 \in \mathbb{R} \) and \( \alpha_1, \alpha_2, \beta_1, \beta_2, a_1, a_2, b_1, b_2 \) satisfy

\[
(a_1 - p + 1) - \sigma |b_1| > 0 \quad \text{and} \quad \sigma(|b_1| - |\beta_1|) - \alpha_1 > 0, \quad (4.11)
\]

\[
\sigma(a_2 - q + 1) - |b_2| > 0 \quad \text{and} \quad (|b_2| - |\beta_2|) - \sigma \alpha_2 > 0, \quad (4.12)
\]

for some constant \( \sigma > 0 \). Then, we have the following result:

**Theorem 4.2.** Set

\[
\gamma_1 = \frac{p(q - 1 - a_2) + qb_1}{(a_1 - p + 1)(a_2 - q + 1) - b_1 b_2}, \quad \gamma_2 = \frac{q(p - 1 - a_1) + pb_2}{(a_1 - p + 1)(a_2 - q + 1) - b_1 b_2}
\]

and assume that

\[
p(q - 1 - a_2) + qb_1 < 0 \quad \text{and} \quad (\alpha_1 - a_1) \gamma_1 - b_1 \gamma_2 > 0, \quad (4.13)
\]
\[ q(p - 1 - a_1) + pb_2 < 0 \quad \text{and} \quad (\alpha_2 - a_2)\gamma_2 - b_2\gamma_1 > 0. \quad (4.14) \]

Then, problem \((P)\) possesses a large solution \((u^*, v^*) \in C_{1,\beta}^1(\Omega) \times C_{1,\beta}^1(\Omega)\), for some \(0 < \beta < 1\), that satisfies the following estimates:

\[ u^*(x) \sim d(x)^{\gamma_1} \quad \text{and} \quad v^*(x) \sim d(x)^{\gamma_2} \quad \text{in} \Omega. \quad (4.15) \]

**Proof.** It suffices to use the same strategy as in the proof of Theorem (3.5). Then under assumptions (4.11) to (4.14), it is also always possible to construct a sub and supersolutions pair to \((P)\) shaped as (4.8), for \(m\) small enough. \(\square\)

### A Proof of Proposition 3.3

We adapt the computations which have already been made in [18] to the case of boundary blow-up solutions. In this section, for sake of simplicity, the eigenvalue \(\lambda_{1,x}\) and the eigenfunction \(\varphi_{1,x}\) defined in the introduction are respectively merely denoted by \(\lambda_1\) and \(\varphi_1\).

**Case 1 : \(\lambda < r\).** For \(t \in (0, \eta]\) we define a Karamata function \(\Theta \in K_\infty\) defined as follows:

\[ \Theta(t) \overset{\text{def}}{=} \exp \left( \int_t^\eta \frac{y(s)}{s} ds \right), \quad (A.1) \]

with \(y \in C_1([0, \eta]) \cap C((0, \eta])\) such that \(y(0) = 0\) and \(\lim_{t \to 0^+} ty'(t) = 0\), carefully chosen. Let \(\beta < 0\). Therefore, from (3.6) in Remark 3.1, we get that

\[
\lim_{\varepsilon \to 0^+} \left( \sup_{x \in \Omega} \frac{\varepsilon \varphi_1(x) \Theta'(\varepsilon \varphi_1(x))}{\Theta(\varepsilon \varphi_1(x))} \right) = \lim_{\varepsilon \to 0^+} \left( \sup_{x \in \Omega} \frac{(\varepsilon \varphi_1(x))^2 \Theta''(\varepsilon \varphi_1(x))}{\Theta(\varepsilon \varphi_1(x))} \right) = 0
\]

Hence, there exists \(\varepsilon > 0\) (small enough) such that: \(\forall x \in \Omega,\)

\[ -\frac{\beta}{2} \leq -\beta - \frac{\varphi_1(x) \Theta'(\varepsilon \varphi_1(x))}{\Theta(\varepsilon \varphi_1(x))} \leq -\frac{3\beta}{2} \]

and

\[ \frac{\beta(\beta - 1)}{2} \leq \beta(\beta - 1) + 2\beta \frac{\varphi_1(x) \Theta'(\varepsilon \varphi_1(x))}{\Theta(\varepsilon \varphi_1(x))} + \frac{\varphi_1(x)^2 \Theta''(\varepsilon \varphi_1(x))}{\Theta(\varepsilon \varphi_1(x))} \leq \frac{3}{2} \beta(\beta - 1). \]

Taking into account of this above consideration, let us define

\[ w(x) \overset{\text{def}}{=} \varphi_1(x)^\beta \Theta(\varepsilon \varphi_1(x)) \quad \text{in} \Omega. \quad (A.2) \]
Then, a direct computation gives us
\[
\Delta_r w = \left( \Theta(\varepsilon \varphi_1) \right)^{r-1} \varphi_1^{(\beta-1)(r-1)-1} \left( -\beta - \frac{\varepsilon \varphi_1 \Theta'(\varepsilon \varphi_1)}{\Theta(\varepsilon \varphi_1)} \right)^{r-2} \times \\
\left[ \lambda_1 \varphi_1^r \left( -\beta - \frac{\varepsilon \varphi_1 \Theta'(\varepsilon \varphi_1)}{\Theta(\varepsilon \varphi_1)} \right) + (r-1) |\nabla \varphi_1|^r \left( \beta(\beta-1) + 2\beta \frac{\varepsilon \varphi_1 \Theta'(\varepsilon \varphi_1)}{\Theta(\varepsilon \varphi_1)} + \frac{(\varepsilon \varphi_1)^2 \Theta''(\varepsilon \varphi_1)}{\Theta(\varepsilon \varphi_1)} \right) \right]
\]
Therefore, from the two last inequalities, (1.6) and (3.7), we get
\[
\Delta_r w(x) \approx \Theta(\varphi_1(x))^{r-1} \varphi_1(x)^{(\beta-1)(r-1)-1} \quad \text{in } \Omega.
\]
On the other hand, from (3.7), we also have
\[
K(x) w(x)^\delta \approx L(\varphi_1(x)) \Theta(\varphi_1(x))^{\delta \varphi_1(x)\beta-\lambda} \quad \text{in } \Omega.
\]
Hence, since \( \delta > r - 1 \), if we choose \( \beta \overset{\text{def}}{=} \frac{r-\lambda}{r-1-\delta} < 0 \) and \( y(t) \overset{\text{def}}{=} \frac{z(t)}{r-1-\delta} \) for \( t \in [0, \eta] \), \( w \) both satisfies \( w|_{\partial \Omega} = +\infty \) and
\[
\Delta_r w(x) \approx K(x) w(x)^\delta \quad \text{in } \Omega. \tag{A.3}
\]
Next, we define
\[
w = mw \quad \text{and} \quad \bar{w} = m^{-1}w \quad \text{in } \Omega,
\]
with a suitable \( 0 < m < 1 \). Using previous estimate (A.3) and taking into account of the super-homogeneous nature (with respect to \( r \)) of problem (Q), it is easy to see that for \( m \) small enough, \( w, \bar{w} \) are respectively a sub and a supersolution to (Q) in \( \Omega \). And finally, applying a classic sub and supersolution method in the interior of \( \Omega \) (as in the proof of Theorem 2.1), we get the existence of a weak solution \( w \in \mathcal{C}_{\text{loc}}^{1,\beta'}(\Omega) \), for some \( 0 < \beta' < 1 \), to (Q), satisfying
\[
w \leq w \leq \bar{w} \quad \text{in } \Omega.
\]

Case 2 : \( \lambda = r \). Similarly to Case 1, for a suitable \( \varepsilon > 0 \) (small enough), we define for \( x \in \Omega \),
\[
w(x) \overset{\text{def}}{=} \Theta(\varepsilon \varphi_1(x)) \quad \text{in } \Omega;
\]
that is to say with \( \beta = 0 \) in (A.2). Once again, we look for a suitable \( \Theta \in \mathcal{K} \) in order to satisfy (A.3), but this time we furthermore assume that \( \Theta \) has to be decreasing in \( (0, \eta] \). Then, let us remark in particular that, from (1.5) and the mean value theorem, we also have
\[
-\Theta'(\varepsilon \varphi_1(x)) \approx -\Theta'(\varphi_1(x)) \quad \text{in } \Omega. \tag{A.4}
\]
Then, in this case we get

$$\Delta_r w = \varphi_1^{-1} \left( -\varepsilon \Theta'(\varepsilon \varphi_1) \right)^{-r-1} \left[ \lambda_1 \varphi_1^r - (r - 1) |\nabla \varphi_1|^r \frac{\varepsilon \varphi_1 \Theta''(\varepsilon \varphi_1)}{\Theta'(\varepsilon \varphi_1)} \right].$$

Let us claim, for the moment, that there exists

$$\lim_{\varepsilon \to 0^+} \left( \sup_{x \in \Omega} \frac{\varepsilon \varphi_1(x) \Theta''(\varepsilon \varphi_1(x))}{\Theta'(\varepsilon \varphi_1(x))} \right) \in (-\infty, 0).$$

Hence, on one hand, together with (1.6) and (A.4), we get that

$$\Delta_r w(x) \approx \varphi_1(x)^{-1} \left( -\Theta'(\varphi_1(x)) \right)^{-r-1} \quad \text{in } \Omega.$$  

And on the other hand, from (3.7), we also have

$$K(x)w(x)^\delta \approx L(\varphi_1(x))\Theta(\varphi_1(x))^\delta \varphi_1(x)^{-r} \quad \text{in } \Omega.$$ 

Then, to get (A.3), it suffices that $\Theta \in K$ satisfies

$$(-\Theta'(t))^{-r-1} \approx t^{1-r} L(t) \Theta(t)^\delta \quad \text{in } (0, \eta].$$

Then, under assumption (3.10), let us define for $t \in (0, \eta]$,

$$\Theta(t) \overset{\text{def}}{=} \left( \int_0^t s^{-1} L(s)^{\frac{1}{r-1}} \, ds \right)^{\frac{r-1}{r-1-\delta}}.$$ 

First, by definition of $L$, we remark that $\Theta$ is a positive and decreasing function of $C^2((0, \eta])$ which satisfies (3.6). Hence, from the characterization of Karamata's functions recalled in point c. of Remark 3.1, it follows that $\Theta \in K$. Secondly, a direct computation ensures us that $\Theta$ satisfies (A.6). Thirdly, an other direct computation gives us

$$\frac{\varepsilon \varphi_1(x) \Theta''(\varepsilon \varphi_1(x))}{\Theta'(\varepsilon \varphi_1(x))} = \frac{1}{r-1} \left( \frac{\varepsilon \varphi_1(x) L'(\varepsilon \varphi_1(x))}{L(\varepsilon \varphi_1(x))} \right) + \frac{\delta}{r-1} \left( \frac{\varepsilon \varphi_1(x) \Theta'(\varepsilon \varphi_1(x))}{\Theta(\varepsilon \varphi_1(x))} \right) - 1$$

Then, taking into account that $L, \Theta \in K$ satisfy (3.6), claim (A.5) holds. And finally, estimate (A.3) also holds for this suitable choice of function $\Theta$ and the rest of the proof is similar to the first case.
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